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ON A DYNAMIC MANAGEMENT DECENTRALIZATION PROBLEM*

A.N. ERMOLOV

A differential game with a fixed termination instant is examined, in which the control on the first player's side is effected by many persons, called agents. Information exchange between agents is not possible. A solution method is suggested, which relies on the investigation of a many-criterion differential game by means of the programmed iterations method. A solution of a linear problem is given for a polyhedral target set and under certain constraints on the control set.

1. **Problem statement.** A differential game is examined, in which the first player (with a control u at his disposal) is a collective of n agents each of whom chooses a control $u_i[t]$ and who, during the operation, observes the position of a vector $x_i[t]$ which depends upon the i-th agent's control but is independent of the controls of the other agents constituting the first player. It is not possible for the agents to exchange information during the game. The agents' goal is to lead the trajectory $x[t]=(x_1[t],\ldots,x_n[t])$ onto a certain target set D at the instant θ, i.e., $x[\theta]\in D$. The given problem generalizes the management decentralization principle studied in /1,2/ to the case of a dynamic system.

Problem 1. Let a collection of subsystems exist, whose motion is described by the ordinary differential equations

$$\begin{aligned}
&x_i^{\cdot}=f_i(t,x_i,u_i,v_i) \\
&t\in[t_0,\theta],\ x_i\in R^{r_i},\ u_i\in P_i(t)\subset R^{p_i} \\
&v_i\in Q_i(t)\subset R^{q_i},\ i=1,\ldots,n \\
&P_i^*\overset{\text{def}}{=}\bigcup_{[t_0,\theta]}P_i(t),\qquad Q_i^*\overset{\text{def}}{=}\bigcup_{[t_0,\theta]}Q_i(t)
\end{aligned}\tag{1.1}$$

where P_i^*, Q_i^* are compacta in R^{p_i} and R^{q_i}, respectively, $i=1,\ldots,n$. The function $f_i:[t_0,\theta]\times R^{r_i}\times P_i^*\times Q_i^*\to R^{r_i}$ is continuous on the whole domain and for every compactum $K\subset R^{r_i}$ and $t_1<t_2$ satisfies a Lipschitz condition with some constant $\Lambda_i(K,t_1,t_2)$ on the set $[t_1,t_2]\times K\times P_i^*\times Q_i^*$; in addition, for any two numbers $t_1<t_2$ there exists $a_i(t_1,t_2)$ such that

$$\|f_i(t,x_i,u_i,v_i)\|_{r_i}\leqslant a_i(t_1,t_2)(\|x_i\|_{r_i}+1)$$

on set $[t_1,t_2]\times R^{r_i}\times P_i^*\times Q_i^*$; $i=1,\ldots,n$. Here $P_i(t)$, $Q_i(t)$ are compact-valued upper-semicontinuous many-valued mappings, integrable with respect to the Lebesgue measure λ on $[t_0,\theta]$ /3/, from R^1 into $2^{R^{p_i}}$ and $2^{R^{q_i}}$, respectively, $i=1,\ldots,n$. The target set

$$D=\left\{x\in R^r\,\Big|\,\sum_{i=1}^n d_i(x_i)\geqslant m^\circ\right\}\tag{1.2}$$

$$r=\sum_{i=1}^n r_i,\quad x=(x_1,\ldots,x_n)\in R^r,\quad m^\circ\in R^\xi,\quad \xi\in N\overset{\text{def}}{=}\{1,2,\ldots\}$$

is prescribed as well. Here $d_i:R^{r_i}\to R^\xi$ is a vector-valued function continuous on R^{r_i}.

To solve Problem 1 we examine an auxiliary many-criterion differential game.

Problem 2.

$$\begin{aligned}
&x^{\cdot}=f(t,x,u,v) \\
&t\in[t_0,\theta],\ x\in R^l,\ u\in P(t)\subset R^p \\
&v\in Q(t)\subset R^q
\end{aligned}$$

The function f and the many-valued mappings $P(t)$, $Q(t)$ satisfy all the conditions satisfied by f_i, $P_i(t)$, $Q_i(t)$ in Problem 1. The first player's goal is to maximize $d(x[\theta])$, where d is a vector-valued function from R^l into R^ξ, continuous on the whole domain.

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For each point (t_*, x_*), $t_* \in [t_0, \theta]$, $x_* \in R^l$, we construct a set $M(t_*, x_*)$ of $m \in R^\xi$, such that the Problem 2 of encounter with set $\{x \in R^l \mid d(x) \geqslant m\}$ at instant θ is solvable from the point (t_*, x_*) as from an initial point. The method of finding $M(t_*, x_*)$ is described below in Theorems 2 and 3. To solve Problem 1 it is necessary to substitute for each i-th subsystem a Problem 2 wherein the role of the vector-valued target function is played by the d_i from (1.2), and to find for it the corresponding set $M_i(t_*, x_*)$. Next, we construct the set

$$M^*(t_*, x_*) = \sum_{i=1}^{n} M_i(t_*, x_{*i})$$

(see Lemma 1 below), where $x_* = (x_{*1}, \ldots, x_{*n})$. By $W = \{(t_*, x_*) \in R^{r+1} \mid m^0 \in M^*(t_*, x_*)\}$ we denote the set of all positions (t_*, x_*) from which, as from initial positions, Problem 1 is solvable for the first player. In order to obtain the collection of strategies of the agents constituting the first player, which solve Problem 1 on encounter with set (1.2) from the initial position $(t_*, x_*) \in W$, it is sufficient to find the collection of vectors $m_i \in M_i(t_*, x_{*i}) \subset R^\xi$, $i = 1, \ldots, n$, such that

$$\sum_{i=1}^{n} m_i \geqslant m^\circ$$

and to solve n differential games with equations of motion coinciding with the equations of motion of the i-th subsystem of (1.1), and with the target set $\{x \in R^{r_i} \mid d^i(x) \geqslant m_i\}$, $i = 1, \ldots, n$.

To solve Problem 2 we use an idea advanced in /4/. Certain notation needed below was introduced in /4,5/ and, therefore, we merely present a verbal description. Let U_v be the first player's counterstrategy in Problem 2; $\{U_v\}$ be the set of all counterstrategies in Problem 2; $\{U_v\}_i$ be the set of all counterstrategies of the i-th agent in Problem 1, $i = 1, \ldots, n$; $X(\cdot) = X(\cdot, t_*, x_*, U_v)$ be the set of all motions generated by the counterstrategy $U_v \in \{U_v\}$ from the point $(t_*, x_*) \in R^{l+1}$ in Problem 2; $X_i(\cdot) = X_i(\cdot, t_*, x_{*i}, U_{vi})$ be the set of motions generated by the counterstrategy $U_{vi} \in \{U_v\}_i$ from the point $(t_*, x_{*i}) \in R^{r_i+1}$ in Problem 1, $i = 1, \ldots, n$. We denote

$$RA \overset{\text{def}}{=} A + \{y \in R^\alpha \mid y \leqslant 0\}, \text{ where } A \subset R^\alpha, \quad \alpha \in N; \quad Ry \overset{\text{def}}{=} R\{y\}.$$

Definition 1. A set $A \subset R^\alpha$ is called an R-set if $RA = A$.

Properties of operator R and of R-sets.

1°. A, $B \subset R^\alpha$, $RA + B = RA + RB = R(A + B)$. If A is an R-set, then $A + B = A + RB = R(A + B)$.

2°. Let $A_z \subset R^\alpha$, $z \in Z$, be a family of R-sets; then $\bigcup_Z A_z$, $\bigcap_Z A_z$ are R-sets.

3°. Let $A_z \subset R^\alpha$, $z \in Z$, be a family of R-sets; then

$$\bigcap_Z (A_z + S_\alpha(0, a)) \subset \bigcap_Z A_z + S_\alpha(0, a\sqrt{\alpha})$$

$$S_\alpha(y, a) = \{x \in R^\alpha \mid \|x - y\|_\alpha \leqslant a\}$$

In Problem 2 we denote

$$M(t_*, x_*) = \bigcup_{\{U_v\}} \bigcap_{X(\theta)} Rd(x)$$

and in Problem 1 we denote the corresponding sets

$$M_i(t_*, x_{*i}) = \bigcup_{\{U_v\}_i} \bigcap_{X_i(\theta)} Rd_i(x_i)$$

for the i-th subsystem. Additionally, we denote

$$M^*(t_*, x_*) = \bigcup_{U_{vi}} \bigcap_{X_i} R\left(\sum_{i=1}^{n} d_i(x_i)\right)$$

where the union is taken over $U_{vi} \in \{U_v\}_i$, $i = 1, \ldots, n$, and the intersection, over $x_i \in X_i(\theta, t_*, x_{*i}, U_{vi})$, $i = 1, \ldots, n$, $x_* = (x_{*1}, \ldots, x_{*n})$.

Lemma 1.

$$M^*(t_*, x_*) = \sum_{i=1}^{n} M_i(t_{*i}, x_{*i})$$

The proof relies on the definition and the properties of the motions /6/.

2. Determination of the set of guaranteed results in Problem 2. Below we require the notation introduced in /4/: $\{E_\lambda, [t_1, t_2]\}$ is the set of second player's programmed controls on $[t_1, t_2]$ in Problem 2; $\{H_\lambda, [t_1, t_2]\}$ is the set of admissible programmed controls on $[t_1, t_2]$ in Problem 2; $\varphi(t)$ is a programmed motion: an absolutely continuous function defined uniquely for each programmed control $\eta \in \{H_\lambda, [t_1, t_2]\}$; $\{\Pi(v), [t_1, t_2]\}$ is the program corresponding to the second player's programmed control $\nu \in \{E_\lambda, [t_1, t_2]\}$; $G(t, t_1, x_1, \nu)$ is the attainability domain at instant $t \in [t_1, t_2]$ for the program $\{\Pi(\nu), [t_1, t_2]\}$, $\nu \in \{E_\lambda, [t_1, t_2]\}$ from the position $(t_1, x_1) \in R^{l+1}$. In /4,5/ all notation was introduced for the case of $P(t)$ and $Q(t)$ not depending on time; however, under the assumptions made all the necessary properties of the programmed constructions are preserved. Let $F(t, x)$ be the many-valued mapping $R^{l+1} \to 2^{R^\xi}$. By $\Gamma(F)$ we denote the many-valued mapping $R^{l+1} \to 2^{R^\xi}$ for which

$$\Gamma(F)(t_*, x_*) = \bigcap_\tau \bigcap_\nu \bigcup_x F(\tau, x)$$

where the union is taken over $\tau \in [t_*, \theta]$ and over $\nu \in \{E_\lambda, [t_*, \tau]\}$ and the intersection, over $x \in G(\tau, t_*, x_*, \nu)$.

Definition 2. The many-valued mapping $C(y): R^\alpha \to 2^{R^\beta}$, $\alpha, \beta \in N$, is said to be uniformly continuous on set $Y \subset R^\alpha$ if $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall y_j \in Y$, $j = 1,2$, $\| y_1 - y_2 \|_\alpha < \delta$, is fulfilled $C(y_j) \subset C(y_{3-j}) + S_\beta(0, \varepsilon)$, $j = 1,2$.

Below we require the following lemmas, presented without proof.

Lemma 2. Let $C(y)$ be a many-valued mapping $R^\alpha \to 2^{R^\beta}$, continuous on the compactum $Y \subset R^\alpha$; then $C(y)$ is uniformly continuous on Y.

Lemma 3. Let $C(y)$ satisfy the conditions of Lemma 2; then $\forall \varepsilon > 0$ $\exists \delta > 0$ for every compactum $K_j \subset Y$, $j = 1,2$, such that

$$\mathrm{dist}^\alpha(K_1, K_2) < \delta, \quad \bigcup_{K_j} C(y) \subset \bigcup_{K_{3-j}} C(y) + S_\beta(0, \varepsilon), \quad j = 1, 2$$

Here $\mathrm{dist}^\alpha(K_1, K_2)$ is the Hausdorff distance in 2^{R^α}, $\alpha \in N$.

Definition 3. The many-valued mapping $C: Y \to 2^{R^\beta}$, $Y \subset R^\alpha$, is said to be R-valued if $C(y)$ is an R-set, $y \in Y$.

Lemma 4. Let $F(t, x): [t_0, \theta] \times R^l \to 2^{R^\xi}$ be an R-valued mapping continuous on the whole domain; then $\Gamma(F)(t, x)$ too is an R-valued mapping on $[t_0, \theta] \times R^l$.

Proof. By virtue of the properties of function f, $\forall (t_*, x_*) \in R^{l+1}$ $\exists \eta \in \{H_\lambda, [t_*, \theta]\}$ $\forall t \in [t_*, \theta]$ $\| \varphi(t, t_*, x_*, \eta) \|_l \leqslant \omega$ (see /4/). Let the sequence $\{(t_k, x_k)\}$ converge to (t_*, x_*) from the right, i.e., $x_k \to x_*$, $t_k \downarrow t_*$. Then

$$\begin{gathered} \forall \gamma > 0 \ \exists k^1 \in N \ \forall k \geqslant k^1 \\ \Gamma(F)(t_k, x_k) \subset \Gamma(F)(t_*, x_*) + S_\xi(0, \gamma) \end{gathered} \tag{2.1}$$

$$\begin{gathered} \forall (\gamma) > 0 \ \exists k^2 \in N \ \ \forall k \geqslant k^2 \\ \Gamma(F)(t_*, x_*) \subset \Gamma(F)(t_k, x_k) + S_\xi(0, \gamma) \end{gathered} \tag{2.2}$$

Let the sequence $\{(t_k, x_k)\}$ converge to (t_*, x_*) from the left. Then for every $\gamma > 0$ we can find $k^3 \in N$ such that relation (2.1) is fulfilled. In addition, for any $\gamma > 0$ there exists $k^4 \in N$ such that (2.2) is fulfilled. The continuity of the many-valued mapping $\Gamma(F)$ follows from the assertions made, while its R-valuedness follows from property 2° of operator R. The proofs of all the assertions are alike; as an example, we present the proof of the first one.

By Lemma 2, $F(t, x)$ is uniformly continuous on $[t_*, \theta] \times S_l(0, \omega)$, i.e.,

$$\begin{gathered} \forall \gamma > 0 \ \exists \varepsilon > 0 \ \forall (t_j, x_j) \in S_{l+1}((t_*, x_*), \varepsilon), \ j = 1, 2 \\ F(t_j, x_j) \subset F(t_{3-j}, x_{3-j}) + S_\xi(0, \gamma/2\sqrt{\xi}) \end{gathered} \tag{2.3}$$

Because function f satisfies a Lipschitz condition on every compactum, $\exists \delta > 0$ $\forall \tau \in [t_*, t_* + \delta]$ $\forall \nu \in \{E_\lambda, [t_*, \theta]\}$ $G(\tau, t_*, x_*, \nu) \subset S_{l+1}((t_*, x_*), \varepsilon)$ follows from (2.3) for $\varepsilon > 0$. Thus, $\forall \gamma > 0$ $\exists \delta > 0$ such that

$$\bigcap_\nu \bigcup_x F(t_* + \delta, x) \subset \bigcup_{(t, x)} F(t, x)$$

where $\nu \in \{E_\lambda, [t_*, \theta]\}$, $x \in G(t_* + \delta, t_*, x_*, \nu)$ and $(t, x) \in S_{l+1}((t_*, x_*), \varepsilon)$. Further, using property 3° of operator R, we obtain

$$\bigcup_{(t, x)} F(t, x) \subset \bigcap_\tau \bigcap_\nu \bigcup_x \left(F(\tau, x) + S_\xi\left(0, \frac{\gamma}{2\sqrt{\xi}}\right)\right) \subset \bigcap_\tau \bigcap_\nu \bigcup_x F(\tau, x) + S_\xi\left(0, \frac{\gamma}{2}\right)$$

where the intersection is taken over $\tau \in [t_*, t_* + \delta]$, $\nu \in \{E_\lambda, [t_*, \theta]\}$, while the union, over $x \in G(\tau, t_*, x_*, \nu)$. We fix $\gamma > 0$. For it we choose a δ and we consider $\tau \in [t_* + \delta, \theta]$. Beginning with some $k_0 \in N$, $t_k \leqslant t_* + \delta \leqslant \tau$, we consider $\nu^k \in \{E_\lambda, [t_k, \theta]\}$ and $\nu^* \in \{E_\lambda, [t_*, \theta]\}$ coinciding with ν^k on $[t_k, \theta]$ /4/. The well-known inequality

$$\operatorname{dist}^l(G(\tau, t_k, x_k, v^k),\ G(\tau, t_*, x_*, v^*)) \leqslant (\|x_k - x_*\|_l + k\omega\,(t_k - t_*)) \cdot \exp(\Lambda_\omega \cdot (\theta - t_*)) \qquad (2.4)$$

$$\Lambda_\omega = \Lambda(S_l(0, \omega), t_0, \theta),\ k_\omega = \max_{(t, x, u, v)} \|f(t, x, u, v)\|_l$$

the maximum is taken over $[t_0, \theta] \times S_l(0, \omega) \times P^* \times Q^*$, is valid for such v^* and v^k. By Lemma 3, $\forall \gamma > 0, \exists \varepsilon > 0$ such that

$$\bigcup_{G_k} F(\tau, x) \subset \bigcup_{G_*} F(\tau, x) + S_\xi\left(0, \frac{\gamma}{2\sqrt{\xi}}\right) \qquad (2.5)$$

if $\operatorname{dist}^l(G_k, G_*) < \varepsilon$; here $G_k = G(\tau, t_k, x_k, v^k)$, $G_* = G(\tau, t_*, x_*, v^*)$. Thus, from estimate (2.4) it follows that $\forall \gamma > 0\ \exists k^1 \geqslant k_0\ \forall \tau \in [t_*, t_* + \delta]\ \forall v^k \in \{E_\lambda, [t_k, \theta]\}$, and relation (2.5) is fulfilled for every $v^* \in \{E_\lambda, [t_*, \theta]\}$ coinciding with v^k on $[t_k, \theta]$. Consequently, using property 3° of operator R, $\forall \gamma > 0\ \exists k^1 \in N\ \ \forall k \geqslant k^1$

$$\Gamma(F)(t_k, x_k) = \bigcap_{[t_k, \theta]} \bigcap_{v^k} \bigcup_{G_k} F(\tau, x) \subset \bigcap_{[t_* + \delta, \theta]} \bigcap_{v^*} \bigcup_{G_*} \left(F(\tau, x) + S_\xi\left(0, \frac{\gamma}{2\sqrt{\xi}}\right)\right) \subset$$

$$\bigcap_{[t_*, \theta]} \bigcap_{v^*} \bigcup_{G_*} F(\tau, x) + S_\xi\left(0, \frac{\gamma}{2}\right) + S_\xi\left(0, \frac{\gamma}{2}\right) = \Gamma(F) + S_\xi(0, \gamma)$$

where $v^k \in \{E_\lambda, [t_k, \theta]\}$, $v^* \in \{E_\lambda, [t_*, \theta]\}$, which proves the first assertion.

We denote

$$M^{(0)}(t_*, x_*) = \bigcap_{v^*} \bigcup_{x} Rd(x)$$

$$v^* \in \{E_\lambda, [t_*, \theta]\}, \quad x \in G(\theta, t_*, x_*, v)$$

$$M^{(k+1)}(t_*, x_*) = \Gamma(M^{(k)})(t_*, x_*),\ k \in N_0 \overset{\text{def}}{=} \{0, 1, \ldots\}$$

Lemma 5. $M^{(0)}(t_*, x_*)$ is an R-valued mapping continuous on $[t_0, \theta] \times R^l$.
The proof is analogous to that of Lemma 4.

Corollary 1. $M^{(k)}(t_*, x_*)$ $k \in N$, is an R-valued mapping continuous on $[t_0, \theta] \times R^l$.

Note 1. $M^{(k+1)}(t_*, x_*) \subset M^{(k)}(t_*, x_*)$, $k \in N_0$.

Later on, in Theorem 1 we prove the convergence of the sequence of sets $M^{(k)}(t, x)$ to the set $M(t, x)$. In meaning, the set $M^{(k)}(t, x)$ consists of all m such that the first player can be guaranteed a payoff m in the many-criterion game in the case when it is possible for the second player to obtain information on the object's phase coordinates no more than k times during the motion.

Theorem 1.

$$\bigcap_{k=0}^{\infty} M^{(k)}(t, x) = M(t, x) \quad t \in [t_0, \theta], \quad x \in R^l$$

Proof. Let us prove that

$$\bigcap_{k=0}^{\infty} M^{(k)}(t, x) \subset M(t, x)$$

Let $m \in M^{(k)}(t_*, x_*)$, $k \in N_0$. $m \in M(t_*, x_*)$ if and only if $(t_*, x_*) \in W_m$, where W_m is a maximal u-stable bridge for Problem 2° with target set $D_m = \{x \in R^l \mid d(x) \geqslant m\}$ /5/. We denote

$$V_m^{(k)} = \{(t, x) \in R^{l+1} \mid m \in M^{(k)}(t, x)\}, \quad k \in N_0$$

$$V_m = \bigcap_{k=0}^{\infty} V_m^{(k)}$$

The set $V_m^{(k)}$ breaks at instant θ on set D_m, i.e., $V_m^{(k)} \cap \{(t, x) \in R^{l+1} \mid t = \theta\} = D_m$, $k \in N_0$ and, consequently, V_m breaks at instant θ on D_m. Thus, it remains to show the u-stability of set V_m /6/. To do this we show that for every position from V_m and for every programmed control of the second player we can find a program confining the motion to set V_m. Let $(t_*, x_*) \in V_m$, $t^* \in [t_*, \theta]$, $v^* \in \{E_\lambda, [t_*, t^*]\}$. Then

$$m \in M^{(k+1)}(t_*, x_*) \subset \bigcup_{G(t^*, t_*, x_*, v^*)} M^{(k)}(t^*, x), \quad k \in N_0$$

Consequently, $x^* \in G(t^*, t_*, x_*, v^*)$ exists such that $m \in M^{(k)}(t^*, x^*)$, and, thus, $\eta^k \in \{\Pi(v^*), [t_*, t^*]\}$: $(t^*, \varphi(t^*, t_*, x_*, \eta^k)) \in V_m^{(k)}$, $k \in N_0$ exists. From the sequence $\{\eta^k\} \subset \{\Pi(v^*), [t_*, t^*]\}$ we pick out a subsequence $*$-weakly converging to some $\eta^* \in \{\Pi(v^*), [t_*, t^*]\}$. The corresponding subsequence $\{\varphi(\cdot, t_*, x_*, \eta^{k_j})\}$ converges uniformly to $\varphi(\cdot, t_*, x_*, \eta^*)$ /5,6/; then $\varphi(t^*, t_*, x_*, \eta^{k_j}) \to \varphi(t^*, t_*, x_*, \eta^*)$ as $j \to \infty$. By virtue of continuity of $M^{(k)}(t, x)$, $V_m^{(k)}$ is a closed set, and from Note 1 it follows that

$$V_m^{(k_j+1)} \subset V_m^{(k_j)},\ \forall j \in N \varphi(t^*, t_* x_* \eta^{k_j}) \in V_m^{(k_j)}$$

whence $\varphi(t^*, t_*, x_*, \eta^*) \in V_m$, but this precisely signifies the u-stability of set V_m.
Analogously /4/ it can be proved that

$$\bigcap_{k=0}^{\infty} M^{(k)}(t,x) \supset M(t,x)$$

3. Linear case of Problem 2. We can examine a linear many-criterion game, i.e., in Problem 2

$$f(t, x, u, v) = A(t)x + B(t)u + C(t)v + g(t)$$

where $A(t)$, $B(t)$, $C(t)$ are matrices of appropriate dimensions, depending continuously on t on $[t_0, \theta]$, $g(t)$ is a continuous vector-valued function; $d(x) = Lx$, where L is a matrix of dimension $\xi \times l$. Since a linear problem can be transformed by a standard procedure (see /6/, for instance), without loss of generality we can reakon that Problem 2 in the linear case appears as follows.

Problem 3.

$$x^\cdot = u + v + g(t);\ u \in P(t) \subset R^l,\ v \in Q(t) \in R^l$$

The function $f(t, x, u, v) = u + v + g(t)$ and the many-valued mappings $P(t)$ and $Q(t)$ satisfy all the conditions of Problem 2. The target function Lx does not change under the transformation.
In what follows we need the set-theoretic operation of geometric difference

$$A \overset{*}{-} B = \bigcap_B (A + B),\ A,\ B \subset R^\alpha,\ \alpha \in N$$

introduced in /7,8/. We present some of its properties.

1°. $(A + h) \overset{*}{-} B = A \overset{*}{-} (B + h) = A \overset{*}{-} B + h$, where $h \subset R^\alpha$, $\alpha \in N$

2°. Let $\psi > 0$, then $(\psi A) \overset{*}{-} (\psi B) = \psi (A \overset{*}{-} B)$

3°. $(A \overset{*}{-} B) \overset{*}{-} C = A \overset{*}{-} (B + C)$

4°. $(A \overset{*}{-} B) + C \subset (A + C) \overset{*}{-} B$

5°. $A \overset{*}{-} C \subset B \overset{*}{-} C$, if $A \subset B$

6°. $A \overset{*}{-} B$ is convex, if A is convex.

7°. Let A, B, C be convex sets from $R^\alpha, \psi \geqslant 0, \chi > 0$. Then $[(C + \psi A) \overset{*}{-} \psi B + \chi A] \overset{*}{-} \chi B = [C + (\psi + \chi) A] \overset{*}{-} (\psi + \chi) B$.

Using property 1° of operator R, for Problem 3 we find

$$M^{(0)}(t_*, x_*) = \bigcap_y RL\Big(x_* + \int_{[t_*, \theta]} P(t)\, d\lambda + y + \int_{[t_*, \theta]} g(t)\, d\lambda\Big) =$$
$$\Big(RL \int_{[t_*, \theta]} P(t)\, d\lambda\Big) \overset{*}{-} \Big(L \int_{[t_*, \theta]} Q(t)\, d\lambda\Big) + L \int_{[t_*, \theta]} g(t)\, d\lambda + Lx_*$$

(the intersection is taken over $y \in \int_{[t_*\theta]} Q(t)\, d\lambda$).

Lemma 6. For Problem 3

$$M^{(k)}(t_*, x_*) = M^{(k)}(t_*, 0) + Lx_*,\ k \in N_0$$

Corollary 2.

$$M(t_*, x_*) = M(t_*, 0) + Lx_*$$

We denote $M_*^{(k)}(t_*, \theta) = M^{(k)}(t_*, 0)$

Note 2. For Problem 3

$$\Gamma(M^{(k)})(t_*, x_*) = \bigcap_{[t_*, \theta]} \Big[\Big(M_*^{(k)}(\tau, \theta) + L \int_{[t_*, \tau]} P(t)\, d\lambda\Big) \overset{*}{-}$$
$$L \int_{[t_*, \tau]} Q(t)\, d\lambda + L \int_{[t_*, \tau]} g(t)\, d\lambda\Big] + Lx_*$$

Theorem 2. In Problem 3 let $P(t) = \mu(t) P + p(t)$ and $Q(t) = \mu(t) Q + q(t)$, where P, Q are compacta in $R^l \mu: [t_0, \theta] \to R^1$, $\mu(t) > 0$, $p(t)$, $q(t)$ are integrable vector-valued functions from $[t_0, \theta]$ into R^l. Then

$$M(t_*, x_*) = M^{(0)}(t_*, x_*) = \int_{[t_*, \theta]} \mu(t)\, d\lambda \cdot (RL \operatorname{conv} P \overset{*}{-}$$

$$L\,\mathrm{conv}\,Q) + L \int_{[t_*,\,\theta]} (p(t) + q(t) + g(t))\,d\lambda + Lx_*$$

Proof. Relying on Note 2, property 1° of operator R and properties 1° and 4° of geometric difference, we obtain

$$\begin{aligned} &M_*^{(1)}(t_*,\theta) = \Gamma(M^{(0)})(t_*,0) = \\ &\bigcap_{[t_*,\theta]} \Big[(M_*^{(0)}(\tau,\theta) + L\int_{[t_*,\tau]} P(t)\,d\lambda) \overset{*}{-} L\int_{[t_*,\tau]} Q(t) + L\int_{[t_*,\tau]} g(t)\,d\lambda\Big] \supset \\ &\bigcap_{[t_*,\theta]} \Big[M_*^{(0)}(\tau,\theta) + \Big(RL\int_{[t_*,\tau]} P(t)\,d\lambda \overset{*}{-} \int_{[t_*,\tau]} Q(t)\,d\lambda\Big) + \\ &L\int_{[t_*,\tau]} g(t)\,d\lambda\Big] = \bigcap_{[t_*,\theta]} (M_*^{(0)}(\tau,\theta) + M_*^{(0)}(t_*,\tau)) \end{aligned}$$

Thus, according to Note 1, from the relation

$$M_*^{(0)}(t_*,\ \tau) + M_*^{(0)}(\tau,\ \theta) \supset M_*^{(0)}(t_*,\ \theta),\quad t_* \in [t_0,\ \theta],\quad \tau \in [t_*,\theta] \tag{3.1}$$

it follows that $M_*^{(1)}(t_*,\theta) = M_*^{(0)}(t_*,\theta)$, $t_* \in [t_0,\theta]$. The latter singifies that

$$M^{(k)}(t,x) = M^{(0)}(t,x),\quad t \in [t_0,\theta],\ x \in R^l,\ k \in N$$

and, since all the hypotheses of Theorem 1 are fulfilled, then $M(t,x) = M^{(0)}(t,x)$.

We verify the validity of (3.1); let $t_0 < t_1 < t_2 < \theta$. Relying on the properties of an integral of a many-valued mapping /3,9/ and on properties 1° and 2° from Sect.3, we obtain

$$\begin{aligned} &M_*^{(0)}(t_1,t_2) = RL\Big(\int_{[t_1,t_2]} \mu(t)\,d\lambda\cdot\mathrm{conv}\,P + \int_{[t_1,t_2]} p(t)\,d\lambda\Big) \overset{*}{-} \\ &L\Big(\int_{[t_1,t_2]} \mu(t)\,d\lambda\cdot\mathrm{conv}\,Q + \int_{[t_1,t_2]} q(t)\,d\lambda\Big) + L\int_{[t_1,t_2]} g(t)\,d\lambda = \\ &\int_{[t_1,t_2]} \mu(t)\,d\lambda\cdot(RL\,\mathrm{conv}\,P \overset{*}{-} L\,\mathrm{conv}\,Q) + \\ &L\int_{[t_1,t_2]} (p(t) + q(t) + g(t))\,d\lambda \end{aligned} \tag{3.2}$$

By property 5° of geometric difference, RL conv $P \overset{*}{-} L$ conv Q is convex; consequently, (3.1) is valid. Having $t_1 = t_*$, $t_2 = \theta$ in (3.2), we complete the theorem's prpof.

We introduce the notation

$$\begin{aligned} &M_\tau^{(0)}(t_*) = M_*^{(0)}(t_*,\theta) \\ &M_\tau^{(k+1)}(t_*) = (M_*^{(k)}(\tau,\theta) + L\int_{[t_*,\tau]} P(t)\,d\lambda\Big) \overset{*}{-} L\int_{[t_1,\tau]} Q(t)\,d\lambda,\quad k \in N_0 \end{aligned}$$

Then for Problem 3

$$M_*^{(k)}(t_*,\theta) = \bigcap_{[t_*,\theta]} M_\tau^{(k)}(t_*)$$

Theorem 3. Let a sequence $\{T_k\}\,\theta = T_0 > T_1 > \ldots > T_k > \ldots$ be specified such that $P(t) = \mu(t)P^k + p(t), Q(t) = \mu(t)Q^{(k)} + q(t),\ t \in [T_k, T_{k-1}]$, where P^k, Q^k are compacta in $R^l, k \in N$, $p(t)$, $q(t)$, $\mu(t)$ satisfy the hypotheses of Theorem 2. Then

$$\begin{aligned} &M(t_*,x_*) = M^{(k)}(t_*,x_*) = M_{T_k}^{(k)}(t_*) + L\int_{[t_*,\theta]} g(t)\,d\lambda + Lx_* \\ &t_* \in [T_{k+1}, T_k],\quad k \in N_0 \end{aligned}$$

Proof. To simplify the calculations the proof is carried out only in the case

$$g(t) = p(t) = q(t) = 0,\ \mu(t) = 1,\ t \in [t_0,\theta] \tag{3.3}$$

According to Lemma 6 and Corollary 2 it sufficies to prove that

$$\begin{aligned} &M(t_*,0) = M_*^{(k)}(t_*,\theta) = M_{T_k}^{(k)}(t_*) \\ &t_* \in [T_{k+1},\ T_k],\ k \in N_0 \end{aligned} \tag{3.4}$$

The proof is by induction: for $k = 0$ the validity of (3.4) was shown in Theorem 2. For $k = 1$ the proof is analogous to that of the induction step. Let (3.4) be true for all $k' = 0, 1, \ldots, k$.

Let $t_* \in [T_{k+2}, T_{k+1}]$, and let us prove

$$M_\tau^{(k+1)}(t_*) \supset M_{T_{k+1}}^{(k+1)}(t_*), \quad \tau \in [t_*, \theta] \tag{3.5}$$

Let $T_{k+1} < \tau < T_k$; relying on the induction hypothesis, the properties of an integral of a many-valued mapping and the properties 3°, 4°, 6°, 7° of geometric difference, we obtain

$$\begin{aligned}
&M_\tau^{(k+1)}(t_*) = \Big(M_*^{(k)}(\tau, \theta) + L \int\limits_{[t_*, \tau]} P(t)\, d\lambda\Big) \overset{*}{-} L \int\limits_{[t_*, \tau]} Q(t)\, d\lambda \supset \\
&\Big\{[(M_*^{(k-1)}(T_k, \theta) + (T_k - \tau) L \operatorname{conv} P^k) \overset{*}{-} (T_k - \tau) L \operatorname{conv} Q^k + \\
&(\tau - T_{k+1}) L \operatorname{conv} P^k] \overset{*}{-} (\tau - T_{k+1}) L \operatorname{conv} Q^k + L \int\limits_{[t_*, T_{k+1}]} P(t)\, d\lambda\Big\} \overset{*}{-} \\
&L \int\limits_{[t_*, T_{k+1}]} Q(t)\, d\lambda = [(M_*^{(k-1)}(T_k, \theta) + (T_k - T_{k+1}) L \operatorname{conv} P^k) \overset{*}{-} \\
&(T_k - T_{k+1}) L \operatorname{conv} Q^k + L \int\limits_{[t_*, T_{k+1}]} P(t)\, d\lambda\Big] \overset{*}{-} L \int\limits_{[t_*, T_{k+1}]} Q(t)\, d\lambda = \\
&\Big(M_*^{(k)}(T_{k+1}, \theta) + L \int\limits_{[t_*, T_{k+1}]} P(t)\, d\lambda\Big) \overset{*}{-} L \int\limits_{[t_*, T_{k+1}]} Q(t)\, d\lambda = M_{T_{k+1}}^{(k+1)}(t_*)
\end{aligned}$$

The validity of (3.5) in case $t_* \leqslant \tau \leqslant T_{k+1}$ and $T_k < \tau \leqslant \theta$ is proved similarly. Consequently,

$$M_*^{(k+1)}(t_*, \theta) = M_{T_{k+1}}^{(k+1)}(t_*) \tag{3.6}$$

Let us prove that

$$M_*^{(k+2)}(t_*, \theta) \supset M_{T_{k+1}}^{(k+1)}(t_*) \tag{3.7}$$

For this it is enough to verify the inclusion

$$M_\tau^{(k+2)}(t_*, \theta) \supset M_{T_{k+1}}^{(k+1)}(t_*)$$

Let $t_* \leqslant \tau \leqslant T_{k+1}$. Relying on the properties of an integral of a many-valued mapping and property 7° of geometric difference, we obtain

$$\begin{aligned}
&M_\tau^{(k+2)}(t_*) = \Big(M_{T_{k+1}}^{(k+1)}(\tau) + L \int\limits_{[t_*, \tau]} P(t)\, d\lambda\Big) \overset{*}{-} L \int\limits_{[t_*, \tau]} Q(t)\, d\lambda = \\
&[(M_*^{(k)}(T_{k+1}, \theta) + (T_{k+1} - \tau) L \operatorname{conv} P^{k+1}) \overset{*}{-} (T_{k+1} - \tau) L \operatorname{conv} Q^k + \\
&(\tau - t_*) L \operatorname{conv} P^{k+1}] \overset{*}{-} (\tau - t_*) L \operatorname{conv} Q^{k+1} = M_{T_{k+1}}^{(k+1)}(t_*)
\end{aligned}$$

When $T_{k+1} < \tau \leqslant \theta$ the validity of (3.7) is verified analogously. Thus, $M(t_*, 0) = M_*^{(k+1)}(t_*, \theta)$, which completes the theorem's proof.

Note 3. Lemma 1 and Theorems 2 and 3 yield as well a solution to Problem 1 with the criterion: maximization at instant θ of the function

$$\min_{j=1,\ldots,\xi} (h_j(\langle l_j, x \rangle))$$

where $l_j \in R^r$, $h_j : R^1 \to R^1$ is a monotonically increasing function, $j = 1, \ldots, \xi$.

Note 4. Suppose that in Problem 1 with target set $\{x \in R^r \mid Lx \geqslant m^\circ\}$, f_i, $P_i(t)$, $Q_i(t)$ satisfy the hypotheses of Theorem 2, $i = 1, \ldots, n$, and let for simplicity, that relations (3.3) are fulfilled. Then by Lemma 1

$$M^*(t_*, x_*) = (\theta - t_*) \sum_{i=1}^{n} (RL_i \operatorname{conv} P_i \overset{*}{-} L_i \operatorname{conv} Q_i) + Lx_*$$

If, however, in Problem 1 the management is effected from a single center, then the solution of such a differential game in terms of set $M(t, x)$ is as well obtained by means of Theorem 2. In this case

$$\begin{aligned}
M^{**}(t_*, x_*) &= (\theta - t_*) \Big(RL \operatorname{conv} \prod_{i=1}^{n} P_i \overset{*}{-} L \operatorname{conv} \prod_{i=1}^{n} Q_i\Big) + Lx_* = \\
&(\theta - t_*) \Big(\Big(\sum_{i=1}^{n} RL_i \operatorname{conv} P_i\Big) \overset{*}{-} \Big(\sum_{i=1}^{n} L_i \operatorname{conv} Q_i\Big)\Big) + Lx_*
\end{aligned}$$

Consequently, in general,

$$M^{**}(t_*, x_*) \supset M^*(t_*, x_*), \; t_* \in [t_0, \theta], \; x_* \in R^r$$

in accord with properties 2° and 4° of geometric difference. However, in case $P_i = \psi_i P$, $Q_i = \psi_i Q$, $\psi_i \in R^1$, $i = 1, \ldots, n$, for example, $M^{**} = M^*$, i.e., in this case decentralized management is no worse that centralized management.

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